Lecture 9: Green Correspondence and Subgroup Relations

Goal: Understand the Green correspondence, its setting in block theory and modular representations, and how it facilitates the transfer of indecomposable modules (especially projectives) between a finite group G and its subgroups, notably those containing defect groups.

1. Background and Motivation

In block theory, many modules are constructed and studied via their behavior under induction and restriction from subgroups. However, these functors are not exact, and their images are often not indecomposable.

The *Green correspondence* provides a bijective relationship between certain indecomposable modules of a group G and those of a subgroup H containing a fixed defect group D.

2. Key Definitions

Definition 9.1 (Vertex and Source). Let M be an indecomposable $F[G]$-module. A subgroup $D \leq G$ is a *vertex* of M if:

- M is a direct summand of $\mathrm{Ind}_D^G(S)$ for some $F[D]$-module S,
- and D is minimal with this property (up to conjugacy).

Such an S is called a *source* of M.

Definition 9.2 (Green Correspondent). Let G be a finite group, p a prime, $D \leq H \leq G$ where D is a defect group. The *Green correspondence* gives a bijection between:

$$\left\{ \begin{array}{l} \text{indecomposable } F[G]\text{-modules with vertex } D \\ \text{in a given block} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{indecomposable } F[H]\text{-modules with vertex } D \\ \text{in the corresponding block} \end{array} \right\}.$$

3. Green's Theorem

Theorem 9.3 (Green Correspondence). Let G be a finite group, $H \leq G$, and $D \leq H$ a p-subgroup. Let M be an indecomposable $F[G]$-module with vertex D. Then there exists a unique (up to isomorphism) indecomposable $F[H]$-module N with vertex D such that:

$$\mathrm{Hom}_{F[D]}(N|_D, M|_D) \neq 0.$$

This module N is called the *Green correspondent* of M in H, denoted $\mathrm{Green}_G^H(M)$.

This establishes a bijection:

$$\mathcal{G}_G^H(D) \longleftrightarrow \mathcal{G}_H^G(D),$$

between isomorphism classes of indecomposable modules with vertex D.

4. Block and Source Compatibility

Proposition 9.4. If M lies in block B of $F[G]$, then its Green correspondent N lies in the corresponding Brauer correspondent block b of $F[H]$.

Theorem 9.5. The Green correspondence respects source modules and defect group structures: the sources of M and N are isomorphic as $F[D]$-modules.

5. Behavior Under Induction and Restriction

Observation:

- $\mathrm{Ind}_H^G(N)$ is not necessarily indecomposable,
- But the summand with vertex D corresponds to M,
- Similarly, $M|_H$ contains N as a summand.

This makes Green correspondence a *vertex-preserving* version of induction and restriction.

6. Example

Example 9.6. Let $G = S_4$, $p = 2$, and let $H = N_G(D)$ where $D = \langle(12)(34)\rangle \cong C_2$.
In the principal block:

- One indecomposable projective module in G with vertex D,
- Has a corresponding indecomposable module in H,
- Green correspondence pairs them uniquely.

Using GAP, one can explore projectives in S_4 and identify their Green correspondents via restriction and analysis of vertex subgroups.

7. Counterexamples

Counterexample 9.7. If the vertex subgroup D is not contained in H, the Green correspondence is not defined.

Counterexample 9.8. The Green correspondence does not preserve dimensions, nor does it preserve being projective or simple in general.

8. Summary

In this lecture we studied:

- The notion of vertices and sources for indecomposable modules,
- The Green correspondence between G- and H-modules with common vertex,
- Block and defect compatibility in Green correspondents,
- How this generalizes induction/restriction in a controlled way.

Coming Up in Lecture 10: We conclude with advanced examples, current research directions, and open problems in modular representation theory and computational approaches.